

Physics 5646
Quantum Mechanics B
Problem Set II

Due: Tuesday, Jan 29, 2019

2.1 Recall that the $j=1/2$ matrix representation of $D(R(\hat{j}\phi))$, i.e. the rotation operator for a y -axis rotation through angle ϕ , is

$$e^{-i\sigma_y\phi/2} = \cos\frac{\phi}{2}\mathbb{1} - i\sigma_y\sin\frac{\phi}{2} = \begin{pmatrix} \cos\frac{\phi}{2} & \sin\frac{\phi}{2} \\ -\sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{pmatrix}.$$

Now consider the Hilbert space of two spin-1/2 particles spanned by the states $|\pm, \pm\rangle$.

(a) Compute $D(R(\hat{j}\phi))|+-\rangle$ and $D(R(\hat{j}\phi))|-+\rangle$, expressing your results in the $|\pm, \pm\rangle$ basis.

(b) Using your results from (a), show directly that

$$D(R(\hat{j}\phi))\frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle) = \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle).$$

(c) Now determine

$$D(R(\hat{j}\phi))\frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle),$$

and express your result as a superposition of total angular momentum states $|sm\rangle = |1\ 1\rangle, |1\ 0\rangle$, and $|1\ -1\rangle$.

2.2 Consider the rank-1 spherical tensor representation of the vector $\mathbf{V} = (V_x, V_y, V_z)$,

$$V_{\pm 1}^{(1)} = \mp \frac{V_x \pm iV_y}{\sqrt{2}}, \quad V_0^{(1)} = V_z.$$

(a) Use the expression for the $j = 1$ matrix representation of the x -axis rotation operator

$D(R(\hat{i}\phi)) = e^{-i\frac{J_x}{\hbar}\phi}$ you obtained in Problem 9-3 last semester,

$$e^{-i\frac{J_x}{\hbar}\phi} = \begin{pmatrix} \frac{1}{2}(1 + \cos\phi) & -\frac{i}{\sqrt{2}}\sin\phi & -\frac{1}{2}(1 - \cos\phi) \\ -\frac{i}{\sqrt{2}}\sin\phi & \cos\phi & -\frac{i}{\sqrt{2}}\sin\phi \\ -\frac{1}{2}(1 - \cos\phi) & -\frac{i}{\sqrt{2}}\sin\phi & \frac{1}{2}(1 + \cos\phi) \end{pmatrix},$$

to evaluate

$$\sum_{q'} D_{q'q}^{(1)}(R(\hat{i}\phi)) V_{q'}^{(1)},$$

for $q = -1, 0, 1$. Verify that your results are what you expect from the transformation properties of (V_x, V_y, V_z) under rotation.

(b) Repeat Part (a) for rotations about the z -axis, i.e. evaluate

$$\sum_{q'} D_{q'q}^{(1)}(R(\hat{k}\phi)) V_{q'}^{(1)},$$

for $q = -1, 0, 1$ and verify again that your result agrees with what you expect from the transformation properties of (V_x, V_y, V_z) . [To do this you will need the $j = 1$ matrix representation of the z -axis rotation operator $D(R(\hat{k}\phi)) = e^{-i\frac{J_z}{\hbar}\phi}$, but this should be straightforward to determine.]

(Since any rotation can be carried out through a sequence of rotations, first about the z -axis, then x -axis, then z -axis again, the above results prove that $V_q^{(1)}$ transforms as a rank-1 spherical tensor for arbitrary rotations.)

2.3 Consider two vector operators $\mathbf{U} = (U_x, U_y, U_z)$ and $\mathbf{V} = (V_x, V_y, V_z)$.

- Construct the two rank-1 spherical tensors, $U_q^{(1)}$ and $V_q^{(1)}$, corresponding to the two vector operators \mathbf{U} and \mathbf{V} .
- Build the rank-0 spherical tensor $T_0^{(0)}$ from $U_q^{(1)}$ and $V_q^{(1)}$ and show that $T_0^{(0)} \propto \mathbf{U} \cdot \mathbf{V}$.
- Build the rank-1 spherical tensor $T_q^{(1)}$ from $U_q^{(1)}$ and $V_q^{(1)}$ and show that $T_0^{(1)} \propto (\mathbf{U} \times \mathbf{V})_z$.
- Build the rank-2 spherical tensor $T_q^{(2)}$ from $U_q^{(1)}$ and $V_q^{(1)}$.

2.4 Quadrupole moment expectation values.

- Using the result of Problem 2.3, construct the rank-2 spherical tensor for the case $\mathbf{U} = \mathbf{V} = \hat{\mathbf{r}}$.

(b) The expectation value,

$$Q = \langle \alpha', j, m = j | (2\hat{z}^2 - \hat{x}^2 - \hat{y}^2) | \alpha, j, m = j \rangle,$$

is known as the quadrupole moment. Using the Wigner-Eckart theorem (and your result from Part (a)), evaluate

$$\langle \alpha', jm' | (\hat{x}^2 - \hat{y}^2) | \alpha, jm \rangle,$$

in terms of Q and the appropriate Clebsch-Gordan coefficients.