Physics 5646
Quantum Mechanics B
Problem Set IX

Due: Thursday, Apr 11, 2019. **DEADLINE EXTENDED TO FRIDAY, Apr 12, 2019**

9.1 Compute the momentum space wave function for the ground state of hydrogen, $\tilde{\psi}_{100}(\vec{p}) = \langle\vec{p}|100\rangle$, where $\langle\vec{r}|100\rangle = \psi_{100}(\vec{r}) = \left(\frac{1}{\pi a_0^3}\right)^{1/2} e^{-r/a_0}$.

Ans:

$$\tilde{\psi}_{100}(\vec{p}) = \left(\frac{1}{2\pi\hbar}\right)^{3/2} \left(\frac{1}{\pi a_0^3}\right)^{1/2} \frac{8\pi/a_0}{\left(a_0^{-2} + (p/\hbar)^2\right)^2}.$$

9.2 [This is a modified version of Parts (a) and (b) of Problem 6.1 in Sakurai.] Consider one-dimensional quantum scattering from a finite range potential, $V(x) \neq 0$ for $-a < x < a$ only.

Suppose the incident state $|k\rangle$ corresponds to an incoming plane wave from the left with wave vector $k > 0$: $\langle x|k\rangle = \frac{1}{\sqrt{2\pi}} e^{ikx}$. As in our three-dimensional analysis, a formal solution to the full Schrödinger equation which reduces to $|k\rangle$ in the limit $V \to 0$ is,

$$|\psi\rangle = |k\rangle + \frac{1}{E - H_0 + i\epsilon} V|\psi\rangle,$$

where it is understood that we take the limit $\epsilon \to 0^+$. (As in the 3D case, we will see this is the correct choice for this scattering problem.) Here $E = \frac{\hbar^2 k^2}{2m}$.

(a) Compute the one-dimensional Green's function,

$$G_+(x, x') = \frac{\hbar^2}{2m} \langle x| \frac{1}{E - H_0 + i\epsilon} |x'\rangle.$$

In doing this, be sure to think carefully about how to complete the contour when doing any integration on the complex plane.

Ans:

$$G_+(x, x') = -\frac{i}{2k} e^{ik|x-x'|}.$$

(b) Using G_+, obtain an integral equation for $\psi(x) = \langle x|\psi\rangle$.

Ans:

$$\psi(x) = \frac{1}{\sqrt{2\pi}} e^{ikx} - \frac{im}{\hbar^2 k} \int_{-a}^{a} e^{ik|x-x'|} V(x')\psi(x') dx'. \tag{1}$$

(c) Consider the special case of scattering from an attractive δ-function potential,

$$V(x) = -\alpha\delta(x).$$

For this potential you can solve (1) for $\psi(x)$ exactly. Do so, and use the result to obtain the reflection and transmission coefficients for this problem. Your result should agree with what you obtained last semester in Problem 6.4(b).

9.3 Consider the scattering of a particle of mass m with energy E from the 'soft-sphere' potential,

$$V(\vec{r}) = \begin{cases} V_0, & \text{if } r \leq a, \\ 0, & \text{if } r > a. \end{cases}$$

(a) Find the scattering amplitude $f(\theta)$ and the differential cross section, $\frac{d\sigma}{d\Omega}$, for this potential in the first-order Born approximation. For the case $ka = 1$ (where $k = \sqrt{2mE/\hbar^2}$) determine the total cross section, σ.

(b) In the limit $E \to 0$, $f(\theta)$ is independent of θ (as is always the case). For the $f(\theta)$ obtained in Part (a), show that if the energy is nonzero, but still small, $f(\theta) \simeq A + B\cos\theta$. Obtain expressions for A and B and determine the ratio B/A.

9.4 Considering the scattering of a particle of mass m with energy E from a spherical delta-function shell of radius a for which the potential energy is,

$$V(\vec{r}) = \alpha\delta(r - a),$$

where α has units of energy times length.

(a) Find $f^{(1)}(\theta)$, the scattering amplitude for this potential in the low-energy first-order Born approximation.

(b) Find $f^{(2)}(\theta)$, the *second*-order Born approximation to the scattering amplitude, in the low-energy limit.

Useful: First- and second-order Born approximations in $E \to 0$ limit:

$$f^{(1)}(\theta) = -\frac{m}{2\pi\hbar^2}\int V(r')d^3r',$$

$$f^{(2)}(\theta) = \left(\frac{m}{2\pi\hbar^2}\right)^2 \int\int V(r')\frac{1}{|\vec{r'} - \vec{r''}|}V(r'')d^3r'd^3r''.$$