Physics 5646
Quantum Mechanics B
Problem Set I

Due: Tuesday, Jan 21, 2020

1.1 Consider two spin-1/2 particles, with spin operators $\mathbf{S}_1$ and $\mathbf{S}_2$. Apply the operator $\mathbf{S}^2 = (\mathbf{S}_1 + \mathbf{S}_2)^2$ to the the three triplet states,

$$|11\rangle \;=\; |++\rangle, \qquad |10\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle + |-+\rangle\right), \qquad |1\;-1\rangle = |--\rangle,$$

and the singlet state,

$$|00\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right),$$

and verify that in all cases $\mathbf{S}^2|sm\rangle = s(s+1)\hbar^2|sm\rangle$.

Hint: To do this you may find the following identity useful:

$$\mathbf{S_1}\cdot\mathbf{S_2} = S_{1z}S_{2z} + \frac{1}{2}\left(S_{1+}S_{2-} + S_{1-}S_{2+}\right).$$

1.2 Calculate the Clebsch-Gordan coefficients for

(a) $1 \otimes \frac{1}{2} = \frac{1}{2} \oplus \frac{3}{2}$, and

(b) $1 \otimes 1 = 0 \oplus 1 \oplus 2$.

In doing this, you may use the property that

$$\langle j_1 m_1, j_2 m_2 | j, m\rangle = (-1)^{j_1+j_2-j}\langle j_1(-m_1), j_2(-m_2)|j(-m)\rangle$$

to reduce the number of coefficients you need to explicitly calculate.

1.3 Argue that $\frac{1}{2}\otimes\frac{1}{2}\otimes\frac{1}{2} = \frac{1}{2}\oplus\frac{1}{2}\oplus\frac{3}{2}$, and verify that the dimensionality of the Hilbert space is the same on both sides of this equation.

1.4 Obtain the Clebsch-Gordan coefficients for $j_1 \otimes \frac{1}{2} = \left(j_1 - \frac{1}{2}\right) \oplus \left(j_1 + \frac{1}{2}\right)$ for arbitrary $j_1 \neq 0$. You can do this by determining the coefficients A and B in

$$|jm\rangle = A|j_1(m+1/2), \frac{1}{2}\;-\frac{1}{2}\rangle + B|j_1(m-1/2), \frac{1}{2}\;\frac{1}{2}\rangle,$$

for which $|jm\rangle$ is a normalized eigenstate of $\mathbf{J}^2 = (\mathbf{J}_1 + \mathbf{J}_2)^2$.

Answer: For $j = j_1 \pm \frac{1}{2}$ and $m = -j, \cdots, j$,

$$\begin{aligned} A &= \left\langle j_1 \left(m + \frac{1}{2}\right), \frac{1}{2} \; -\frac{1}{2} \middle| jm \right\rangle = \sqrt{\frac{1}{2} \mp \frac{m}{2j_1 + 1}}, \\ B &= \left\langle j_1 \left(m - \frac{1}{2}\right), \frac{1}{2} \; \frac{1}{2} \middle| jm \right\rangle = \pm\sqrt{\frac{1}{2} \pm \frac{m}{2j_1 + 1}}, \end{aligned}$$

where the phase is chosen to be consistent with our convention.

1.5 An electron in a hydrogen atom is in the state $|\psi\rangle$ with position/spin-space wave function (in spinor notation),

$$\begin{pmatrix} \psi_+(\vec{r}) \\ \psi_-(\vec{r}) \end{pmatrix} = R_{21}(r) \left[\sqrt{\frac{1}{3}} Y_1^0(\theta, \phi) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sqrt{\frac{2}{3}} Y_1^1(\theta, \phi) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right],$$

where $R_{21}(r)$ is the $n = 2$, $l = 1$ hydrogen atom radial function.

(a) What are the possible results of measuring the orbital angular momentum squared, $\mathbf{L}^2$, and what are the probabilities for each possibility?

(b) What are the possible results of measuring z-component of the orbital angular momentum, L_z, and what are the probabilities for each possibility?

(c) What are the possible results of measuring the spin angular momentum squared, $\mathbf{S}^2$, and what are the probabilities for each possibility?

(d) What are the possible results of measuring the z-component of the spin angular momentum, S_z, and what are the probabilities for each possibility?

(e) What are the possible results of measuring the total angular momentum squared, $\mathbf{J}^2 = (\mathbf{L} + \mathbf{S})^2$, and what are the probabilities for each possibility?

(f) What are the possible results of measuring the z-component of the total angular momentum, J_z, and what are the probabilities for each possibility?

(g) If you measure the position of the particle, what is the probability density to find it at the point r, θ, ϕ?

(h) If you measure the z-component of the spin and the distance from the origin, what is the probability density for finding the particle with spin up at a distance r from the origin?

Hint: To do parts (e) and (f) you may find the Clebsch-Gordan coefficients computed in Problem 1.4 helpful.