Physics 5646
Quantum Mechanics B
Problem Set XI

Due: Friday, Apr 24, 2020

11.1 Probability current for Dirac equation.

Show that if the four-component spinor ψ satisfies the Dirac equation,

$$\left[c\vec{\alpha} \cdot \frac{\hbar}{i}\vec{\nabla} + \beta mc^2 + e\phi(\vec{r})\right]\psi = i\hbar\frac{\partial}{\partial t}\psi,$$

where $e\phi(\vec{r})$ is the potential energy of the particle due to the electrostatic potential $\phi(\vec{r})$, then

$$\frac{\partial}{\partial t}(\psi^\dagger\psi) = -\vec{\nabla} \cdot \vec{j},$$

where

$$\vec{j} = c\psi^\dagger\vec{\alpha}\psi.$$

11.2 Non-relativistic limit of Dirac equation.

(a) Prove that if $\vec{\Pi} = \hat{\vec{p}} - \frac{e}{c}\vec{A}$ is the kinematic momentum in the presence of a vector potential $\vec{A}$ that

$$\vec{\Pi} \times \vec{\Pi} = \frac{ie\hbar}{c}\vec{B}$$

where $\vec{B} = \vec{\nabla} \times \vec{A}$. (Hint: This essentially follows from Problem 6.3(b) from earlier this semester.)

(b) Show that for vector potential $\vec{A} = (-yB, xB, 0)/2$ corresponding to a uniform field B pointing in the z-direction, $\vec{\nabla} \times \vec{A} = B\hat{k}$,

$$\frac{(\vec{\Pi} \cdot \vec{\sigma})^2}{2m} = \frac{\hat{\vec{p}}^2}{2m} - \frac{eB}{2mc}(L_z + 2S_z) + O(B^2),$$

where $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$. (Hint: You may use the identity $(\vec{a} \cdot \vec{\sigma})(\vec{b} \cdot \vec{\sigma}) = \vec{a} \cdot \vec{b} + i(\vec{a} \times \vec{b}) \cdot \vec{\sigma}$ which you derived in Problem 8.3 last semester.) Since $\frac{(\vec{\sigma}\cdot\vec{\Pi})^2}{2m}$ is the effective Hamiltonian describing the non-relativistic limit of the Dirac equation, this result shows that

that this equation describes a spin-1/2 particle with magnetic moment $\vec{\mu} = \frac{e}{mc}\vec{S}$, in agreement with experiment.

11.3 Consider solutions to the free particle Dirac equation,

$$\left(c\vec{\alpha} \cdot \hat{\vec{p}} + \beta mc^2\right)\psi = i\hbar\frac{\partial}{\partial t}\psi$$

for which the momentum vector points in the positive z direction.

(a) Show that in this case the state

$$\psi(\vec{r}, t) = ue^{i(pz-Et)/\hbar},$$

where u is a four-component Dirac spinor which does not depend on $\vec{r}$ or t, can be a solution of the Dirac equation provided that u satisfies the equation,

$$\begin{pmatrix} mc^2 & 0 & cp & 0 \\ 0 & mc^2 & 0 & -cp \\ cp & 0 & -mc^2 & 0 \\ 0 & -cp & 0 & -mc^2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = E\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}.$$

(b) Show that there are two eigenstates corresponding to the same positive energy eigenvalue,

$$E = E_p^+ = E_p = \sqrt{(pc)^2 + (mc^2)^2},$$

which can be chosen to be the following

$$u^{(R)} = A\begin{pmatrix} 1 \\ 0 \\ \frac{cp}{E_p+mc^2} \\ 0 \end{pmatrix}, \quad u^{(L)} = B\begin{pmatrix} 0 \\ 1 \\ 0 \\ -\frac{cp}{E_p+mc^2} \end{pmatrix}.$$

Here A and B are normalization constants which you do not need to determine.

(c) In addition, show that there are two eigenstates corresponding to the same negative energy eigenvalues

$$E = E_p^- = -E_p = -\sqrt{(pc)^2 + (mc^2)^2},$$

which can be chosen to be the following

$$v^{(R)} = C \begin{pmatrix} -\frac{cp}{E_p+mc^2} \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad v^{(L)} = D \begin{pmatrix} 0 \\ \frac{cp}{E_p+mc^2} \\ 0 \\ 1 \end{pmatrix}.$$

Again, C and D are normalization constants which you do not need to determine.

(d) Show that the states labeled L and R above are also eigenstates of the helicity operator $\hat{p}_{op} \cdot \vec{\Sigma}$ where

$$\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix},$$

and $\hat{p}_{op} = \frac{\hat{\vec{p}}}{|\hat{\vec{p}}|}$ is the operator corresponding to the unit vector pointing in the direction of $\vec{p}$.