Physics 5646
Quantum Mechanics B
Problem Set II

Due: Tuesday, Jan 28, 2020

2.1 Recall that the j=1/2 matrix representation of $D(R(\hat{j}\phi))$, i.e. the rotation operator for a y-axis rotation through angle ϕ, is

$$e^{-i\sigma_y\phi/2} = \cos\frac{\phi}{2}\mathbb{1} - i\sigma_y \sin\frac{\phi}{2} = \begin{pmatrix} \cos\frac{\phi}{2} & \sin\frac{\phi}{2} \\ -\sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{pmatrix}.$$

Now consider the Hilbert space of two spin-1/2 particles spanned by the states $|\pm,\pm\rangle$.

(a) Compute $D(R(\hat{j}\phi))|+-\rangle$ and $D(R(\hat{j}\phi))|-+\rangle$, expressing your results in the $|\pm,\pm\rangle$ basis.

(b) Using your results from (a), show directly that

$$D(R(\hat{j}\phi))\frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right) = \frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right).$$

(c) Now determine

$$D(R(\hat{j}\phi))\frac{1}{\sqrt{2}}\left(|+-\rangle + |-+\rangle\right),$$

and express your result as a superposition of total angular momentum states $|sm\rangle = |1\ 1\rangle, |1\ 0\rangle$, and $|1\ -1\rangle$.

2.2 Consider the rank-1 spherical tensor representation of the vector $\mathbf{V} = (V_x, V_y, V_z)$,

$$V^{(1)}_{\pm 1} = \mp\frac{V_x \pm iV_y}{\sqrt{2}}, \quad V^{(1)}_0 = V_z.$$

(a) Use the expression for the $j = 1$ matrix representation of the x-axis rotation operator $D(R(\hat{i}\phi)) = e^{-i\frac{J_x}{\hbar}\phi}$ you obtained in Problem 9-3 last semester,

$$e^{-i\frac{J_x^{(1)}}{\hbar}\phi} = \begin{pmatrix} \frac{1}{2}(1+\cos\phi) & -\frac{i}{\sqrt{2}}\sin\phi & -\frac{1}{2}(1-\cos\phi) \\ -\frac{i}{\sqrt{2}}\sin\phi & \cos\phi & -\frac{i}{\sqrt{2}}\sin\phi \\ -\frac{1}{2}(1-\cos\phi) & -\frac{i}{\sqrt{2}}\sin\phi & \frac{1}{2}(1+\cos\phi) \end{pmatrix},$$

to evaluate

$$\sum_{q'} D^{(1)}_{q'q}(R(\hat{\imath}\phi))V^{(1)}_{q'},$$

for $q = -1, 0, 1$. Verify that your results are what you expect from the transformation properties of (V_x, V_y, V_z) under rotation.

(b) Repeat Part (a) for rotations about the z-axis, i.e. evaluate

$$\sum_{q'} D^{(1)}_{q'q}(R(\hat{k}\phi))V^{(1)}_{q'},$$

for $q = -1, 0, 1$ and verify again that your result agrees with what you expect from the transformation properties of (V_x, V_y, V_z). [To do this you will need the $j = 1$ matrix representation of the z-axis rotation operator $D(R(\hat{k}\phi)) = e^{-i\frac{J_z}{\hbar}\phi}$, but this should be straightforward to determine.]

(Since any rotation can be carried out through a sequence of rotations, first about the z-axis, then x-axis, then z-axis again, the above results prove that $V^{(1)}_q$ transforms as a rank-1 spherical tensor for arbitrary rotations.)

2.3 Consider two vector operators $\mathbf{U} = (U_x, U_y, U_z)$ and $\mathbf{V} = (V_x, V_y, V_z)$.

(a) Construct the two rank-1 spherical tensors, $U^{(1)}_q$ and $V^{(1)}_q$, corresponding to the two vector operators $\mathbf{U}$ and $\mathbf{V}$.

(b) Build the rank-0 spherical tensor $T^{(0)}_0$ from $U^{(1)}_q$ and $V^{(1)}_q$ and show that $T^{(0)}_0 \propto \mathbf{U} \cdot \mathbf{V}$.

(c) Build the rank-1 spherical tensor $T^{(1)}_q$ from $U^{(1)}_q$ and $V^{(1)}_q$ and show that $T^{(1)}_0 \propto (\mathbf{U}\times\mathbf{V})_z$.

(d) Build the rank-2 spherical tensor $T^{(2)}_q$ from $U^{(1)}_q$ and $V^{(1)}_q$.

2.4 You are given the following H-atom matrix element,

$$\langle 210|\hat{p}_z|100\rangle = i\frac{16\sqrt{2}}{81}\frac{\hbar}{a_0}.$$

Here $\hat{\mathbf{p}}$ is the momentum operator, a_0 is the Bohr radius, and the standard H-atom eigenstates are labeled $|nlm\rangle$, where n, l, and m are the usual H-atom quantum numbers (ignoring spin).

(a) Using the Wigner-Eckart theorem, determine the matrix element $\langle 211|\hat{p}_y|100\rangle$.

(b) Compute the expectation value of $\hat{p}_y$ in the state $\frac{1}{\sqrt{2}}(|100\rangle + |211\rangle)$.

2.5 Quadrupole moment expectation values.

(a) Using the result of Problem 2.3, construct the rank-2 spherical tensor for the case $\mathbf{U} = \mathbf{V} = \hat{\mathbf{r}}$.

(b) The expectation value,

$$Q = \langle \alpha', j\ m = j|(2\hat{z}^2 - \hat{x}^2 - \hat{y}^2)|\alpha, j\ m = j\rangle,$$

is known as the quadrupole moment. Using the Wigner-Eckart theorem (and your result from Part (a)), evaluate

$$\langle \alpha', jm'|(\hat{x}^2 - \hat{y}^2)|\alpha, jm\rangle,$$

in terms of Q and the appropriate Clebsch-Gordan coefficients.