Physics 5646
Quantum Mechanics B
Problem Set V

Due: Thursday, Feb 20, 2020

5.1 Reconsider the Hamiltonian for a spin-1/2 particle with gyromagnetic ratio γ in a magnetic field $\mathbf{B} = B\hat{\imath} + B_0\hat{k}$ you studied in Problem 4.3,

$$H = -\gamma \mathbf{S} \cdot \mathbf{B} = -\frac{\hbar\gamma}{2}\left(B_0\sigma_z + B\sigma_x\right).$$

Once again treat the term proportional to B as a perturbation, but this time consider the limit $B \gg B_0$ for which the unperturbed levels are "almost" degenerate.

Show that the exact eigenstates of H in this limit closely resemble would you would obtain by taking $B_0 = 0$, so that the unperturbed levels are degenerate, and applying degenerate perturbation theory to this problem.

5.2 Consider an isotropic harmonic oscillator in two dimensions with Hamiltonian,

$$H_0 = \frac{1}{2m}\left(\hat{p}_x^2 + \hat{p}_y^2\right) + \frac{1}{2}m\omega^2(\hat{x}^2 + \hat{y}^2).$$

(a) What are the energies of the three lowest-lying states? Is there any degeneracy?

Now consider the effect of the perturbation

$$V = \lambda m\omega^2\hat{x}\hat{y},$$

on this oscillator. Here λ is a dimensionless real number much smaller than one in magnitude.

(b) Find the zeroth-order energy eigenkets and the corresponding energies to first-order in λ for each of these three low-lying states.

(c) Solve the $H_0 + V$ problem exactly and compare the results you obtain with what you find in Part (b). **Hint:** Try rotating the coordinate system from (x, y) to (x', y') where $x' = \frac{1}{\sqrt{2}}(x + y)$ and $y' = \frac{1}{\sqrt{2}}(-x + y)$.

5.3 Work out the Stark effect to lowest non-vanishing order for the $n = 3$ levels of the hydrogen atom. Obtain not only the first-order energy shifts due to the perturbation $V = -e\mathcal{E}\hat{z}$, but also the corresponding zeroth-order eigenkets. Ignore any fine-structure corrections or spin.

5.4 Consider the effect on the $n = 2$ states of the hydrogen atom due to a perturbation of the form,

$$V = \lambda(\hat{x}^2 - \hat{y}^2).$$

Again ignore any fine-structure corrections or spin.

(a) Recall that $\hat{x}^2 - \hat{y}^2 = T_2^{(2)} + T_{-2}^{(2)}$ where $T_q^{(2)}$ is the spherical tensor of rank 2 you constructed in Part (a) of Problem 2.5. Using this fact, determine which matrix matrix elements of V in the $|2lm\rangle$ basis do not vanish.

(b) Determine the first-order energy shifts of the $n = 2$ states and the corresponding zeroth-order eigenkets using degenerate perturbation theory.

Some H-atom radial functions [needed Y_l^ms can be found in Appendix B of Sakurai and Napolitano (pg. 528)]:

$$\begin{aligned}
R_{10}(r) &= \frac{1}{a_0^{3/2}}\, 2\, e^{-r/a_0}, \\
R_{20}(r) &= \frac{1}{(2a_0)^{3/2}} \left(2 - \frac{r}{a_0}\right) e^{-r/2a_0}, \\
R_{21}(r) &= \frac{1}{(2a_0)^{3/2}} \frac{1}{\sqrt{3}} \frac{r}{a_0}\, e^{-r/2a_0}, \\
R_{30}(r) &= \frac{1}{(3a_0)^{3/2}}\, 2\left(1 - \frac{2}{3}\frac{r}{a_0} + \frac{2}{27}\frac{r^2}{a_0^2}\right) e^{-r/3a_0}, \\
R_{31}(r) &= \frac{1}{(3a_0)^{3/2}} \frac{4\sqrt{2}}{9} \frac{r}{a_0} \left(1 - \frac{1}{6}\frac{r}{a_0}\right) e^{-r/3a_0}, \\
R_{32}(r) &= \frac{1}{(3a_0)^{3/2}} \frac{2\sqrt{2}}{27\sqrt{5}} \frac{r^2}{a_0^2}\, e^{-r/3a_0}
\end{aligned}$$