Physics 5646
Quantum Mechanics B
Problem Set VII

Due: Thursday, Mar 12, 2020

7.1 Projection Theorem.

In what follows you may use the fact that

$$\langle 10, jj|jj\rangle = -\sqrt{\frac{j}{j+1}},$$

(this is a special case of one of the Clebsch-Gordan coefficients I calculated in class) and the Wigner-Eckart theorem, which, using Sakurai's convention for the reduced matrix element, is

$$\langle \alpha', j'm'|T_q^{(k)}|\alpha, jm\rangle = \langle kq, jm|j'm'\rangle \frac{\langle \alpha', j'||T^{(k)}||\alpha, j\rangle}{\sqrt{2j+1}}.$$

(a) Show that

$$\frac{\langle \alpha', j'||J^{(1)}||\alpha, j\rangle}{\sqrt{2j+1}} = -\delta_{\alpha\alpha'}\delta_{jj'}\sqrt{j(j+1)}.$$

(b) Consider an arbitrary vector operator $\vec{V} = (V_x, V_y, V_z)$. Using the fact that

$$\vec{J}\cdot\vec{V} = J_z V_z + \frac{1}{2}\left(J_+ V_- + J_- V_+\right),$$

where $V_\pm = V_x \pm iV_y$, show that

$$\langle \alpha', jm'|\vec{J}\cdot\vec{V}|\alpha, jm\rangle = c_{jm}\frac{\langle \alpha', j||V^{(1)}||\alpha, j\rangle}{\sqrt{2j+1}}.$$

where c_{jm} does not depend on α, α' or $\vec{V}$. Then use the result of Part (a) to show that

$$c_{jm} = -\hbar\sqrt{j(j+1)}\delta_{mm'}.$$

(c) Using the previous result, show that

$$\langle \alpha', jm'|V_q^{(1)}|\alpha, jm\rangle = \frac{\langle \alpha', jm|\vec{J}\cdot\vec{V}|\alpha, jm\rangle}{\hbar^2 j(j+1)}\langle jm'|J_q^{(1)}|jm\rangle.$$

This result is known as the projection theorem.

(d) Use the projection theorem to show that if $|njm; ls\rangle$ is the H-atom state with total angular momentum quantum number j, J_z quantum number m, and orbital and spin quantum numbers l and s $(= 1/2)$, that

$$\langle njm; ls|V_Z|njm; ls\rangle = -\frac{e\hbar}{2m_ec}g_{jls}mB; \quad g_{jls} = 1 + \frac{j(j+1) - l(l+1) + 3/4}{2j(j+1)},$$

where

$$V_z = -\frac{e}{2m_ec}(L_z + 2S_z)B,$$

is the Zeeman term describing the coupling of a magnetic field $\vec{B} = B\hat{k}$ to the orbital and spin magnetic moments of the electron.

7.2 Hyperfine Structure.

In addition to the Coulomb interaction, the electron and proton interact via the so-called hyperfine interaction, In the $1s$ state of the Hydrogen atom, this interaction takes the form

$$H_{hf} = A\vec{S}_e \cdot \vec{S}_p,$$

where $\vec{S}_e$ and $\vec{S}_p$ are the electron and proton spins, respectively.

(a) Show that H_{hf} splits the ground state of the hydrogen atom into two levels with energies,

$$E_+ = E_1 + \frac{\hbar^2 A}{4}, \qquad E_- = E_1 - \frac{3\hbar^2 A}{4},$$

where the electron and proton spins form a singlet (triplet) in the state(s) with energy E_- (E_+). Here E_1 is the ground state energy of hydrogen without H_{hf} ($E_1 = -\frac{e^2}{2a_0}$ in the absence of fine structure corrections).

(b) The origin of the hyperfine coupling is the interaction between the magnetic dipole moments of the electron, $\vec{\mu}_e = -\frac{|e|}{m_ec}\vec{S}_e$, and proton, $\vec{\mu}_p = g_p\frac{|e|}{2m_pc}\vec{S}_p$, where m_e and m_p are electron and proton masses, respectively, and $g_p \simeq 5.7$ is the proton g-factor. While this interaction is somewhat complicated, the net effect in the $1s$ state is an effective interaction of the form

$$H \simeq -\frac{1}{a_0^3}\vec{\mu}_e \cdot \vec{\mu}_p.$$

Using this, estimate the coupling A in Part (a), and show that the splitting between E_+ and E_- is on the order of

$$E_+ - E_- \sim \frac{m_e}{m_p}\alpha^2 \frac{e^2}{2a_0}.$$

Thus we see that the hyperfine splitting, like the fine structure splittings, is on the order of $\alpha^2 E_1$, but is also smaller by an additional factor of m_e/m_p. Using this result, estimate the order of magnitude of the wavelength of a photon emitted when the atom undergoes a transition from E_+ to E_-. [The actual wavelength for this transition is 21 cm — the famous "21-cm line" of great importance in radio astronomy.]

7.3 He Atom: First-Order Ground State Energy Shift.

The Hamiltonian for a He-like atom in which two electrons are bound to a nucleus of charge $Z|e|$ is

$$H = H_0 + H_{12},$$

where

$$H_0 = \frac{\hat{\vec{p}}_1^{\,2}}{2m} - \frac{Ze^2}{\hat{r}_1} + \frac{\hat{\vec{p}}_2^{\,2}}{2m} - \frac{Ze^2}{\hat{r}_2} \quad \text{and} \quad H_{12} = \frac{e^2}{|\hat{\vec{r}}_1 - \hat{\vec{r}}_2|}.$$

Let H_0, which describes two noninteracting electrons in the presence of the charge $+Z|e|$ nucleus, be the unperturbed Hamiltonian, and let $|\psi^0_{GS}\rangle$ denote the unperturbed ground state of H_0, which, in the position representation, is

$$\langle \vec{r}_1, \vec{r}_2, m_{s1}, m_{s2} | \psi^0_{GS}\rangle = \psi_{100,Z}(\vec{r}_1)\psi_{100,Z}(\vec{r}_2)\chi_{singlet}(m_{s1}, m_{s2}),$$

where

$$\psi_{100,Z}(\vec{r}) = \left(\frac{Z^3}{\pi a_0^3}\right)^{1/2} e^{-Zr/a_0}.$$

Calculate that the first order ground state energy shift due to the perturbation H_{12},

$$E^1_{GS} = \langle \psi^0_{GS}|H_{12}|\psi^0_{GS}\rangle = \int |\psi_{100,Z}(\vec{r}_1)|^2 |\psi_{100,Z}(\vec{r}_2)|^2 \frac{e^2}{|\vec{r}_1 - \vec{r}_2|} d^3r_1 d^3r_2.$$

[Answer: $E^1_{GS} = \frac{5}{8}\frac{Ze^2}{a_0}$.]

7.4 Use the following trial wave function with variational parameter $\lambda > 0$,

$$\psi_\lambda(x) = \begin{cases} A_\lambda (x^2 - \lambda^2)^2, & -\lambda < x < \lambda \\ 0, & \text{otherwise} \end{cases}$$

to estimate the ground state energy of the one-dimensional harmonic oscillator with Hamiltonian $H = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2$. Compare your result to the exact ground state energy $\frac{1}{2}\hbar\omega$.

7.5 Use the variational wave function $\psi_\alpha(\vec{r}) = A_\alpha e^{-\alpha r^2}$ to estimate the ground state energy of the hydrogen atom with Hamiltonian $H = \frac{\hat{\vec{p}}^2}{2m} - \frac{e^2}{\hat{r}}$. Compare your variational result to the exact result.