Physics 5646
Quantum Mechanics B
Problem Set VIII

Due: Thursday, Mar 26, 2020

8.1 Rabi Oscillations. [This is essentially Problem 5.30 in Sakurai and Napolitano. The only differences are 1) slight notational changes, and 2) I allow the parameter γ to be complex.] Consider a two-level system with $E_1 < E_2$ and corresponding states $|1\rangle$ and $|2\rangle$. There is a time-dependent potential that connects the two levels as follows,

$$\langle 1|V|1\rangle = \langle 2|V|2\rangle = 0, \quad \langle 1|V|2\rangle = \gamma e^{i\omega t},\ \langle 2|V|1\rangle = \gamma^* e^{-i\omega t}$$

The time-dependent state of the system, in the interaction picture, is $|\psi(t)\rangle_I = c_1(t)|1\rangle + c_2(t)|2\rangle$. It is known that at $t = 0$ the system is in the state $|1\rangle$, so $c_1(0) = 1$ and $c_2(0) = 0$.

(a) Find $|c_1(t)|^2$ and $|c_2(t)|^2$ for $t > 0$ by *exactly* solving the coupled differential equations,

$$\begin{aligned} i\hbar\dot{c}_1 &= \gamma e^{i(\omega-\omega_0)t} c_2, \\ i\hbar\dot{c}_2 &= \gamma^* e^{-i(\omega-\omega_0)t} c_1. \end{aligned}$$

where $\omega_0 = (E_2 - E_1)/\hbar$.

(b) Do the same problem using first-order time-dependent perturbation theory. Compare the two approaches for small values of $|\gamma|$. Treat the following cases separately: (i) ω very different from ω_0 and (ii) ω close to ω_0.

Answer for (a):

$$\begin{aligned} |c_2(t)|^2 &= \frac{|\gamma|^2/\hbar^2}{|\gamma|^2/\hbar^2 + (\omega-\omega_0)^2/4} \sin^2\left\{\left[\frac{|\gamma|^2}{\hbar^2} + \frac{(\omega-\omega_0)^2}{4}\right]^{1/2} t\right\}, \\ |c_1(t)|^2 &= 1 - |c_2(t)|^2. \end{aligned}$$

8.2 [Problem 5.29 in Sakurai and Napolitano, again with minor changes.] Consider a system consisting of two spin-1/2 objects. For $t < 0$ the spins do not interact and the Hamiltonian can be taken to be $H = 0$. For $t > 0$ the Hamiltonian is given by,

$$H = \frac{J}{\hbar^2}\mathbf{S}_1 \cdot \mathbf{S}_2.$$

Suppose the system is in the state $|+-\rangle$ for $t < 0$. Find, as a function of time, the probability for the system to be found in the states $|++\rangle$, $|+-\rangle$, $|-+\rangle$, and $|--\rangle$,

(a) By solving the problem exactly.

(b) By solving the problem assuming the validity of first-order time-dependent perturbation theory with H as the perturbation switched on at $t = 0$. Under what conditions does your perturbation result accurately describe the exact result?

8.3 Consider a one dimensional quantum particle of mass m moving in the presence of the harmonic potential $V(x) = \frac{1}{2}m\omega^2 x^2$. For time $t < 0$ the particle is in its ground state. The particle then experiences the following time-dependent perturbation:

$$V(t) = \begin{cases} 0 & t < 0 \\ \kappa \hat{x}^2 & 0 \leq t \leq t_0 \\ 0 & t > t_0 \end{cases}$$

Assuming that first-order time-dependent perturbation theory is valid, find an expression for the probability that the particle will be found in an excited state for time $t > t_0$.

8.4 Problem 5.22, Sakurai and Napolitano, Pg. 380.

8.5 Problem 5.28, Sakurai and Napolitano, Pg. 382.